First-order differential calculi and Laplacians on q-deformations of compact semisimple Lie groups

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Quantum Groups Seminar

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Laplacian on Riemannian manifolds

- (M, g): n-dimensional Riemannian manifold

Definition

The Laplacian on (M, g) is $\square : C^\infty(M) \to C^\infty(M)$ defined locally by

$$\square f = -\left(\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}\right) + O(g).$$

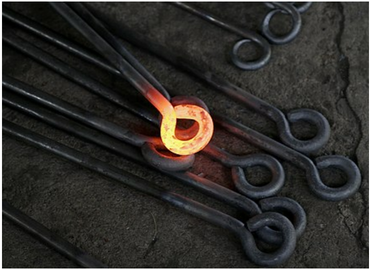
- The heat equation on M:

$$\frac{\partial u}{\partial t}(t, x) = -\square_x u(t, x), \quad u \in C^\infty(\mathbb{R}_{\geq 0} \times M)$$

$\rightsquigarrow \left\{ e^{-t\square} : L^2(M) \to L^2(M) \,\middle|\, t \geq 0 \right\}$: Heat semigroup on (M, g)

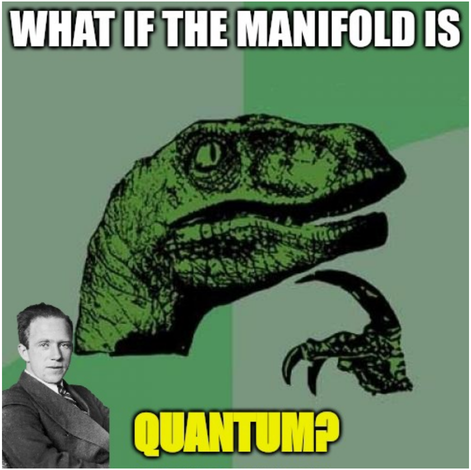
$\rightsquigarrow$ Geometric invariants, e.g., Atiyah-Singer index theorem.

How Laplacian reveals geometry

Manifold → Laplacian → Heat equation → Geometry

Big picture



Noncommutative manifold → Laplacian →
Heat equation → Noncommutative geometry

Question

What is Laplacian on a noncommutative manifold?

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Compact quantum groups

Compact quantum groups afford many well-behaving "noncommutative manifolds".

Compact quantum group

Definition

A unital $*$-algebra $\mathcal{A}$ (over $\mathbb{C}$) equipped with the following structure maps is called a **compact quantum group (CQG)**.

- (Comultiplication) $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$: unital $*$-algebra homomorphism

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xleftarrow[\Delta \otimes \mathrm{id}]{} & \mathcal{A} \otimes \mathcal{A} \\ {\scriptstyle \mathrm{id} \otimes \Delta} \uparrow & & \uparrow {\scriptstyle \Delta} \\ \mathcal{A} \otimes \mathcal{A} & \xleftarrow[\Delta]{} & \mathcal{A} \end{array}$$

- (Counit) $\epsilon : \mathcal{A} \to \mathbb{C}$: $*$-algebra homomorphism

$$\begin{array}{ccccc} \mathcal{A} & \xleftarrow{\mathrm{id} \otimes \epsilon} & \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\epsilon \otimes \mathrm{id}} & \mathcal{A} \\ & {\scriptstyle =} \nwarrow & \uparrow {\scriptstyle \Delta} & \nearrow {\scriptstyle =} & \\ & & \mathcal{A} & & \end{array}$$

Compact quantum group

- (Antipode) $S : \mathcal{A} \to \mathcal{A}$: linear map

$$\begin{array}{ccccc}
\mathcal{A} \otimes \mathcal{A} & \xleftarrow{S \otimes \mathrm{id}} & \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\mathrm{id} \otimes S} & \mathcal{A} \otimes \mathcal{A} \\
\downarrow \mu & & \Delta \uparrow & & \downarrow \mu \\
\mathcal{A} & \xrightarrow{\quad \epsilon \quad} & \mathcal{A} & \xleftarrow{\quad \epsilon \quad} & \mathcal{A}
\end{array}$$

- (Haar state) $h : \mathcal{A} \to \mathbb{C}$: linear functional
  - (Positivity) $h(a^* a) \geq 0$
  - (Invariance) $(h \otimes \mathrm{id})\Delta(a) = h(a)1_{\mathcal{A}} = (\mathrm{id} \otimes h)\Delta(a)$
  - (Faithfulness) $h(a^* a) = 0$ implies $a = 0$.
  - (Normalization) $h(1_{\mathcal{A}}) = 1$

First-order differential calculus on a CQG

- $\mathcal{A}$: CQG
- Ω: $\mathcal{A}$-bimodule
- $\mathcal{A} \xrightarrow{d} \Omega$: $\mathbb{C}$-linear map

Definition (Woronowicz '89)

The pair (Ω, d) is a **first-order differential calculus (FODC) over** $\mathcal{A}$ if

- (Leibniz rule) $d(ab) = (da)b + a(db)$ for all $a, b \in \mathcal{A}$
- (Standard form) Every element $\omega \in \Omega$ can be expressed as

$$\omega = \sum_{j=1}^{k} a_j db_j$$

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Matrix coefficients

K: Compact Hausdorff group

Definition

- (Matrix coefficients) $\pi : K \to \mathrm{End}(V)$: conti. fin. dim'l repn, $v, w \in V$

$$K \ni x \longmapsto \langle v, \pi(x)w \rangle \in \mathbb{C}$$

- $\mathrm{Pol}(K)$: the unital $*$-algebra of matrix coefficients of K

The CQG $\mathrm{Pol}(K)$

$\mathrm{Pol}(K)$ is a CQG when endowed with:

- (Comultiplication) $\Delta(f)(x, y) = f(xy)$
- (Counit) $\epsilon(f) = f(e)$
- (Antipode) $S(f)(x) = f(x^{-1})$
- (Haar state) $\hbar(f) = \int_K f(x)dx$ (integration w.r.t. the Haar measure).

Classical first-order differential calculus

- K: compact Lie group with Lie algebra $\mathfrak{k}$
- $\exp : \mathfrak{k} \to K$: exponential map
- $X \in \mathfrak{k}$ and $f \in \mathrm{Pol}(K) \rightsquigarrow$ Define $Xf \in \mathrm{Pol}(K)$ by

$$Xf(x) = \frac{d}{dt}\Big|_{t=0} f\big(x \exp(tX)\big), \quad x \in K.$$

- $\{X_1, \cdots, X_n\}$: basis of $\mathfrak{k}$

Classical first-order differential calculus

The space of differential 1**-forms** is defined to be the $\mathrm{Pol}(K)$-module

$$\Omega_K = \mathrm{Pol}(K)^n.$$

The classical differential $d : \mathrm{Pol}(K) \to \Omega_K$ is given by

$$df = (X_1 f, \cdots, X_n f).$$

Characterization of the classical Laplacian

- Consider the $\mathrm{Pol}(K)$-valued sesquilinear form on $\Omega_K = \mathrm{Pol}(K)^n$

$$\langle \cdot\,,\,\cdot \rangle : \Omega_K \times \Omega_K \ni \Big((f_j)_{j=1}^n, (g_j)_{j=1}^n\Big) \longmapsto \sum_{j=1}^n \overline{f_j} g_j \in \mathrm{Pol}(K).$$

Characterization of Laplacian

The linear map $\square : \mathrm{Pol}(K) \to \mathrm{Pol}(K)$ defined by

$$\square f = -(X_1^2 f + \cdots + X_n^2 f), \quad f \in \mathrm{Pol}(K)$$

is a **Laplacian** on K. It is *the unique map* satisfying

$$\int_K \overline{f(x)}(\square g)(x)dx = \int_K \langle df, dg\rangle(x)dx, \quad \forall f, g \in \mathrm{Pol}(K).$$

Characterization of Laplacian

- Consider the $\mathrm{Pol}(K)$-valued sesquilinear form on $\Omega_K \cong \mathrm{Pol}(K)^n$

$$\langle \cdot , \cdot \rangle : \Omega_K \times \Omega_K \ni \left((f_j)_{j=1}^n, (g_j)_{j=1}^n \right) \longmapsto \sum_{j=1}^n \overline{f_j} g_j \in \mathrm{Pol}(K).$$

Characterization of Laplacian

The linear map $\square : \mathrm{Pol}(K) \to \mathrm{Pol}(K)$ defined by

$$\square f = -(X_1^2 + \cdots X_n^2) f, \quad f \in \mathrm{Pol}(K)$$

is a **Laplacian** on K. It is *the unique map* satisfying

$$h(f^* \square g) = \int_K \overline{f(x)} (\square g)(x) dx = \int_K \langle df, dg \rangle (x) dx = h(\langle df, dg \rangle).$$

- The red-colored expressions admit generalizations to CQGs!

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Definition of Laplacian on a CQG

- $\mathcal{A}$: CQG
- (Ω, d): FODC over $\mathcal{A}$
- $\Omega \times \Omega \xrightarrow{\langle \cdot, \cdot \rangle} \mathcal{A}$: sesquilinear map such that

$$\Omega \times \Omega \ni (\omega, \eta) \longmapsto h\big(\langle \omega, \eta \rangle\big) \in \mathbb{C}$$

is nondegenerate. We call this property **strong nondegeneracy**.

Definition

A (unique) linear operator $\square : \mathcal{A} \to \mathcal{A}$ satisfying

$$h(f^* \square g) = h\big(\langle df, dg \rangle\big), \quad \forall f, g \in \mathcal{A}$$

is called **the Laplacian associated with** $\big(\Omega, d, \langle \cdot, \cdot \rangle\big)$.

Remark

- **Previous approaches** (Heckenberger et al., Landi et al., Majid et al.)

FODC $(\Omega, d, \langle\cdot,\cdot\rangle) \Rightarrow$ Laplacian: The unique map $\square : \mathcal{A} \to \mathcal{A}$ satisfying

$$h(f^* \square g) = h\big(\langle df, dg\rangle\big)$$

Remark

- **Previous approaches** (Heckenberger et al., Landi et al., Majid et al.)

FODC $(\Omega, d, \langle\cdot,\cdot\rangle) \Rightarrow$ Laplacian: The unique map $\Box : \mathcal{A} \to \mathcal{A}$ satisfying
$$h(f^* \Box g) = h\big(\langle df, dg\rangle\big)$$

- **Our approach**

Operator $\Box : \mathcal{A} \to \mathcal{A} \quad \Longrightarrow \quad$ FODC $(\Omega_\Box, d, \langle\cdot,\cdot\rangle)$

$\Downarrow$

$\Box$ becomes Laplacian:
$$h(f^* \Box g) = h\big(\langle df, dg\rangle\big)$$

Main construction

- $\mathcal{A} = \bigoplus_{\mu \in \mathrm{Irr}(\mathcal{A})} M_{n_\mu}(\mathbb{C})$: Peter-Weyl decomposition of the CQG $\mathcal{A}$

Theorem (L' 24)

Let $\square : \mathcal{A} \to \mathcal{A}$ be a linear map such that

L1 $\square$ diagonalizes over the Peter-Weyl decomposition with real eigenvalues

L2 $\square S = S \square$

L3 $\square(1) = 0$.

Then, $\square$ is a Laplacian, i.e., there exists an FODC $(\Omega, d, \langle\cdot,\cdot\rangle)$ over $\mathcal{A}$ s.t.

$$\hbar(f^* \square g) = \hbar\big(\langle df, dg\rangle\big), \quad \forall f, g \in \mathcal{A}.$$

Main construction

- $\mathcal{A} = \bigoplus_{\mu \in \mathrm{Irr}(\mathcal{A})} M_{n_\mu}(\mathbb{C})$: Peter-Weyl decomposition of the CQG $\mathcal{A}$

Theorem (L' 24)

Let $\Box : \mathcal{A} \to \mathcal{A}$. If (1) $\Box$ diagonalizes over the Peter-Weyl decomposition with real eigenvalues, (2) $\Box S = S\Box$, and (3) $\Box(1) = 0$, then

- $R_\Box = \left\{ f \in \mathrm{Ker}\,\epsilon \,\middle|\, \epsilon\,\Box\,(fg) = 0,\ \forall g \in \mathrm{Ker}\,\epsilon \right\}$ is an ad-invariant right ideal
  $\rightsquigarrow$ we get bicovariant FODC $(\Omega_\Box, d)$
- The map $\langle \cdot, \cdot \rangle : \Omega_\Box \times \Omega_\Box \to \mathcal{A}$ defined by

$$\langle adb, fdg \rangle = -\frac{1}{2}(a\, b_{(1)})^*\, f\, g_{(1)}\ \epsilon \circ \Box\Big(\big(b^*_{(2)} - \epsilon(b^*_{(2)})\big)\big(g_{(2)} - \epsilon(g_{(2)})\big)\Big)$$

  for $a, b, f, g \in \mathcal{A}$ is a **strongly nondegenerate $\mathcal{A}$-sesquilinear form.**
- The linear map $\Box : \mathcal{A} \to \mathcal{A}$ becomes a Laplacian, i.e., it satisfies
  $$h(f^* \Box g) = h\big(\langle df, dg \rangle\big), \quad \forall f, g \in \mathcal{A}$$

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q-deformation

- K: simply-connected compact semisimple Lie group with Lie algebra $\mathfrak{k}$
- $U(\mathfrak{g})$: universal enveloping algebra of $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{k}$
- The bilinear pairing $\mathfrak{k} \times \mathrm{Pol}(K) \ni (X, f) \longmapsto Xf(e) \in \mathbb{C}$ extends to

$$U(\mathfrak{g}) \times \mathrm{Pol}(K) \to \mathbb{C} \xrightarrow{q\text{-deformation}} U_q(\mathfrak{g}) \times \mathrm{Pol}(K_q) \to \mathbb{C}$$

- $U_q(\mathfrak{g})$: **quantized universal enveloping algebra**
- $\mathrm{Pol}(K_q)$: **quantized algebra of functions** $\rightsquigarrow \mathrm{Pol}(K_q)$ is a CQG.

Theorem (L' 24)

- Classification of linear operators on $\mathrm{Pol}(K_q)$ satisfying L1–L3 (and L4)
- For any finite dimensional bicovariant FODC on K_q, there are infinitely many Laplacians associated with it.

Remark

- **Previous approaches** (Heckenberger et al., Landi et al., Majid et al.)

FODC $(\Omega, d, \langle\cdot,\cdot\rangle) \Rightarrow$ Laplacian: The unique map $\Box : \mathcal{A} \to \mathcal{A}$ satisfying
$$h(f^* \Box g) = h\big(\langle df, dg\rangle\big)$$

- **Our approach**

$$\text{Operator } \Box : \mathcal{A} \to \mathcal{A} \quad \Longrightarrow \quad \text{FODC } \big(\Omega_\Box, d, \langle\cdot,\cdot\rangle\big)$$

$$\Downarrow$$

$\Box$ becomes Laplacian:
$$h(f^* \Box g) = h\big(\langle df, dg\rangle\big)$$

Laplacians on K_q

Theorem (L' 24)

Linear operators on $\mathrm{Pol}(K_q)$ satisfying L1 and L4 are given by linear combinations of operators on $\mathrm{Pol}(K_q)$ of the form

$$(\mathrm{Ev}_\zeta\, z_\mu) \triangleright f = (\mathrm{id} \otimes \mathrm{Ev}_\zeta \otimes z_\mu)\Delta^2(f), \quad f \in \mathrm{Pol}(K_q)$$

where Ev_ζ for $\zeta \in Z \subseteq K$, the center of K, is given by

$$\mathrm{Ev}_\zeta : \mathrm{Pol}(K_q) \twoheadrightarrow \mathrm{Pol}(Z) \xrightarrow{\mathrm{ev}_\zeta} \mathbb{C},$$

and $z_\mu \in ZU_q(\mathfrak{g})$ for $\mu \in \mathbf{P}^+$ is **quantum Casimir element**.

Comparison with the classical Laplacian

- $\Box$: classical Laplacian on K
- $\Box_q$: a Laplacian on K_q in the classification

	$\Box : \mathrm{Pol}(K) \to \mathrm{Pol}(K)$	$\Box_q : \mathrm{Pol}(K_q) \to \mathrm{Pol}(K_q)$
Spectrum	discrete, nonnegative	discrete, lower-semibounded
Eigenvalues	diverge to ∞	diverge to ∞
$\hbar(\langle\cdot,\cdot\rangle)$	positive definite	nondegenerate

Table: Comparison with the classical Laplacian

- **Most importantly,** $\lim_{q\to 1} \Box_q = \Box$

The $q \to 1$ limits of q-Laplacians

Theorem (L' 24)

As $q \to 1$, the Laplacian $\Box_q$ converges to the classical Laplacian $\Box$ on K:

$$\begin{array}{ccc}
\mathrm{Pol}(K_q) & \xrightarrow{\quad \Box_q \quad} & \mathrm{Pol}(K_q) \\
\| & & \| \\
\bigoplus_{\lambda \in \mathbf{P}^+} M_{n_\lambda}(\mathbb{C}) & \xrightarrow{\left(c_q(\lambda)\right)_\lambda} & \bigoplus_{\lambda \in \mathbf{P}^+} M_{n_\lambda}(\mathbb{C})
\end{array}$$

as $q \to 1 \downarrow$

$$\begin{array}{ccc}
\bigoplus_{\lambda \in \mathbf{P}^+} M_{n_\lambda}(\mathbb{C}) & \xrightarrow{\left(c(\lambda)\right)_\lambda} & \bigoplus_{\lambda \in \mathbf{P}^+} M_{n_\lambda}(\mathbb{C}) \\
\| & & \| \\
\mathrm{Pol}(K) & \xrightarrow{\quad \Box \quad} & \mathrm{Pol}(K)
\end{array}$$

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The heat kernel and the heat semigroup on K_q

- $C(K_q)$: C^*-algebra completion of $\mathrm{Pol}(K_q)$

Definition (L, Lee, Wang, Youn in progress)

- The family of elements

$$p_t = \sum_{\mu \in \mathbf{P}^+} d_\mu e^{-tc_q(\mu)} \sum_{1 \le i,j \le n_\mu} (F_\mu)_{ij} u^\mu_{ji} \in C(K_q), \quad t > 0$$

is called **the heat kernel on** K_q.

- The family of bounded operators

$$P_t : C(K_q) \ni f \longmapsto \big(\mathrm{id} \otimes h(\,\cdot\, p_t)\big)\Delta(f) \in C(K_q), \quad t > 0$$

and $P_0 = \mathrm{id}_{C(K_q)}$ is called **the heat semigroup on** $C(K_q)$.

* One can define heat semigroups on $L^p(K_q)$ for all $1 \le p \le \infty$ using p_t.

Properties of the heat semigroup

Proposition (L, Lee, Wang, Youn in progress)

- $P_{t_1+t_2} = P_{t_1} P_{t_2}$ for $t_1, t_2 \geq 0$
- For $v \in C(K_q)$, the map $t \mapsto P_t v \in C(K_q)$ is continuous on $(0, \infty)$.

Theorem (L' 24)

The operators P_t are not **completely positive**.

Remark

The heat diffusion processes on K_q are not **quantum Markov processes**.

The heat equation on K_q

Heat equation on K_q (L, Lee, Wang, Youn in progress)

Given $v \in C(K_q)$, does there exist $u : [0, \infty) \to C(K_q)$ s.t.

$$u_0 = v, \quad \frac{d}{dt} u_t + \Box_q u_t = 0?$$

And if it does, is the solution unique?

- We found that $u_t = P_t v$ is a solution.
- However, we failed to prove that $\lim_{t \to 0} P_t v = v$, which holds iff

$$\{P_t \mid 0 \leq t \leq 1\} \text{ is uniformly bounded}$$

by Banach-Steinhaus theorem.

Thank you for your attention